LEAST SQUARES METHODS WITH YULE-WALKER EQUATIONS

- Yule-Walker equations of the form $\begin{bmatrix} \mathbf{R}_B \\ \mathbf{R}_A \end{bmatrix} \mathbf{a} = \begin{bmatrix} \mathbf{c} \\ \mathbf{0} \end{bmatrix}$

  are extended to $\begin{bmatrix} \mathbf{R}_B \\ \mathbf{R}_E \end{bmatrix} \mathbf{a} = \begin{bmatrix} \mathbf{c} \\ \mathbf{0} \end{bmatrix}$

  where

$$\mathbf{R}_E = \begin{bmatrix} R_x[Q+1] & R_x[Q] & \cdots & R_x[Q-P+1] \\ \vdots & \vdots & \ddots & \vdots \\ R_x[L] & R_x[L-1] & \cdots & R_x[L-P] \end{bmatrix}$$

  $L > P + Q$

- Lower block is solved by least squares.

LEAST SQUARES YULE-WALKER

The Yule-Walker equations in theory produce

$$\mathbf{R}_E\,\mathbf{a} = \mathbf{0}$$

but in practice, when estimates are used for $\mathbf{R}_E$,

$$\mathbf{R}_E\,\mathbf{a} = \boldsymbol{\epsilon}$$

Application of least squares results in

$$(\mathbf{R}_E^{*T}\mathbf{R}_E)\mathbf{a} = \begin{bmatrix} \mathcal{S} \\ \mathbf{0} \end{bmatrix}$$

where $\mathcal{S}$ is the sum of squared errors.

SUGGESTED PROCEDURE

Begin with an order higher than necessary:

$$P' > P \quad \text{and} \quad Q' > Q$$

Follow one of two methods:

- Find an order (P', Q') model of reduced rank.
- Find an order (P, Q) model by averaging techiques.

LS YULE-WALKER METHOD 1
(RANK P ORDER (P', Q') MODEL)

1. P is the "effective rank" of $\mathbf{R}_E$: $\left(\frac{\sigma_1^2+\sigma_2^2+\cdots+\sigma_P^2}{\sigma_1^2+\sigma_2^2+\cdots+\sigma_{P'}^2}\right)^{\frac{1}{2}} \approx 1$

2. With $\mathbf{a} = \begin{bmatrix} 1 \\ \\ \mathbf{a}' \end{bmatrix}$ and $\mathbf{R}_E = \begin{bmatrix} | & \\ \mathbf{r}_\circ & \mathbf{R}'_E \\ | & \end{bmatrix}$ $\mathbf{a}'$ is found from

   $\mathbf{a}' = -\mathbf{R}'^{+}_E \mathbf{r}_\circ$

   where terms $1/\sigma_k$ for $k > P$ in $\mathbf{R}'^{+}_E$ are set to zero.

3. MA parameters are found in the usual way.

LS YULE-WALKER METHOD 2
(ORDER (P, Q) MODEL)

1. Form submatrices with $P+1$ columns from $\mathbf{R}_E$.

   Example: $(P' = 2,\ P = 1)$

$$\begin{bmatrix} R_x[3] & R_x[2] & R_x[1] \\ R_x[4] & R_x[3] & R_x[2] \\ R_x[5] & R_x[4] & R_x[3] \\ R_x[6] & R_x[5] & R_x[4] \end{bmatrix};\ \mathbf{R}_{(0)} = \begin{bmatrix} R_x[3] & R_x[2] \\ R_x[4] & R_x[3] \\ R_x[5] & R_x[4] \\ R_x[6] & R_x[5] \end{bmatrix},\ \mathbf{R}_{(1)} = \begin{bmatrix} R_x[2] & R_x[1] \\ R_x[3] & R_x[2] \\ R_x[4] & R_x[3] \\ R_x[5] & R_x[4] \end{bmatrix}$$

2. Solve equations of the form

$$\left(\sum_{k=0}^{P'-P} \mathbf{R}_{(k)}^{*T} \mathbf{R}_{(k)} \right) \mathbf{a} = \begin{bmatrix} \mathcal{S} \\ \mathbf{0} \end{bmatrix}$$

APPLICATION OF METHODS TO DATA

ARMA DATA (Kay, 1988)

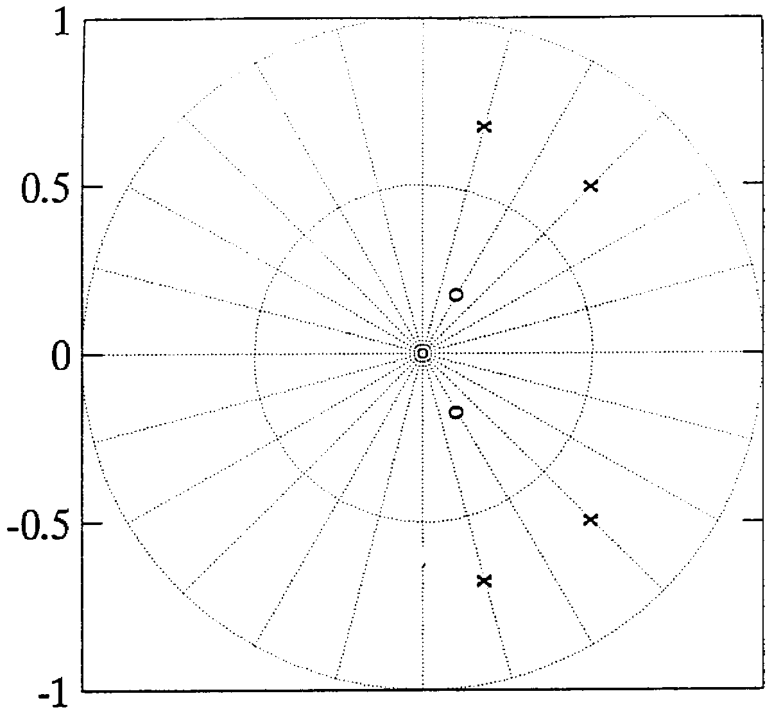
	a_1	a_2	a_3	a_4	b_0	b_1	b_2
ARMA1	−1.352	1.338	−0.662	0.240	1.000	−0.200	0.040
ARMA3	−2.760	3.809	−2.654	0.924	1.000	−0.900	0.810

RECORDED DATA

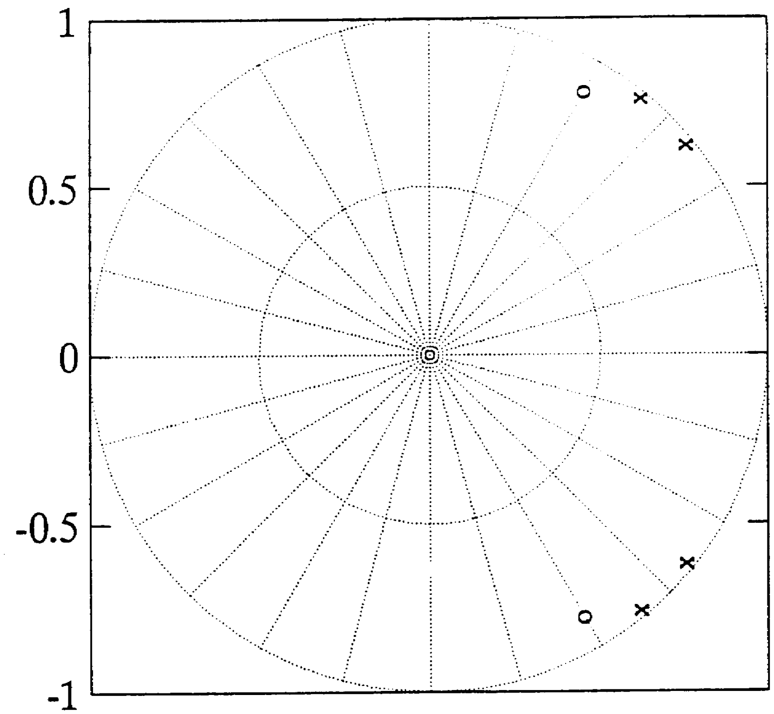
Sound of ruler struck on book.

POLE-ZERO LOCATIONS FOR ARMA DATA

ARMA1

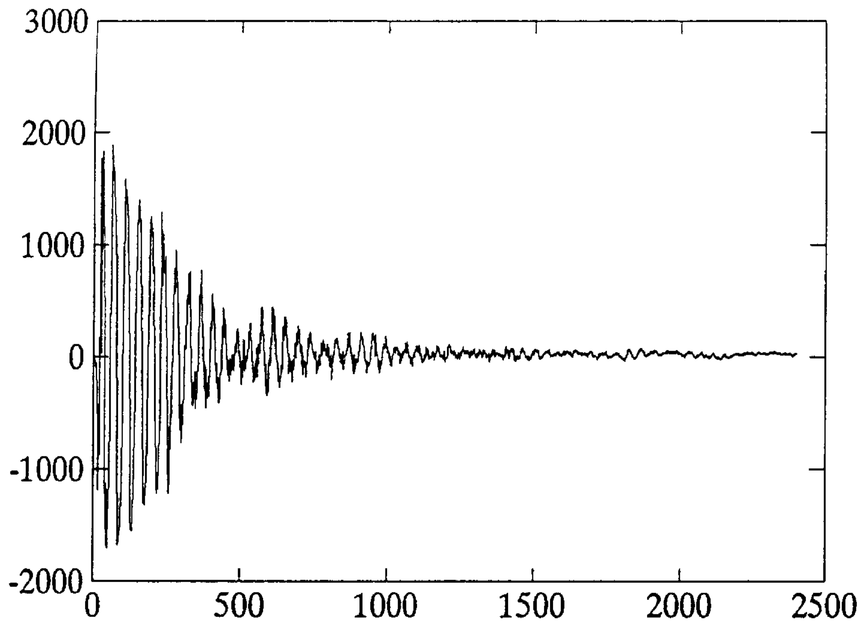


ARMA3

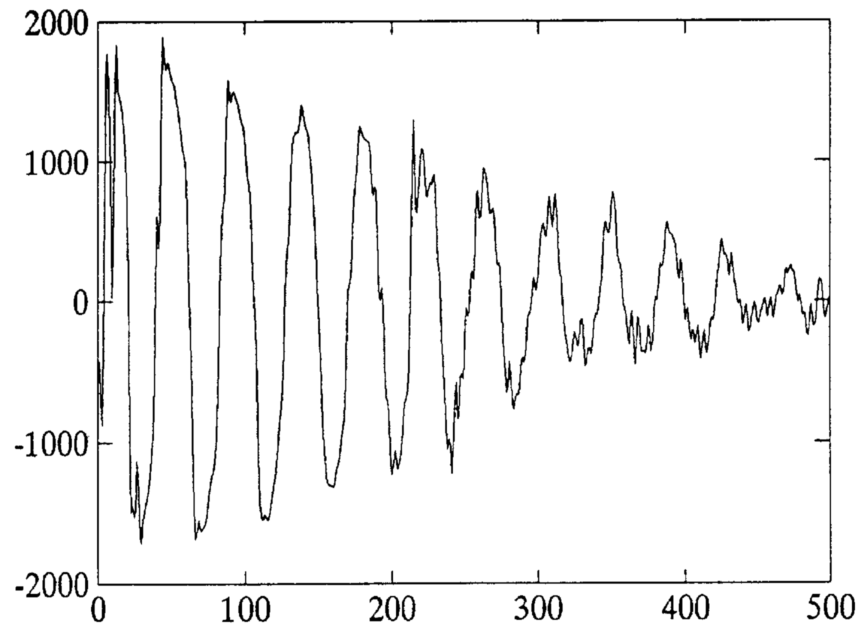


RECORDED RULER DATA

COMPLETE SEQUENCE

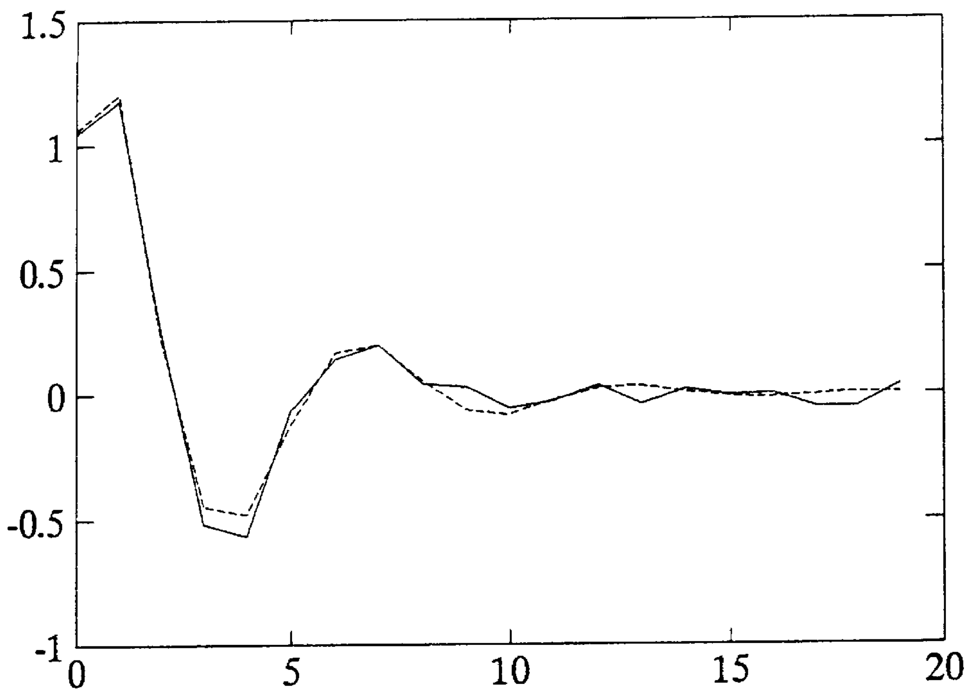


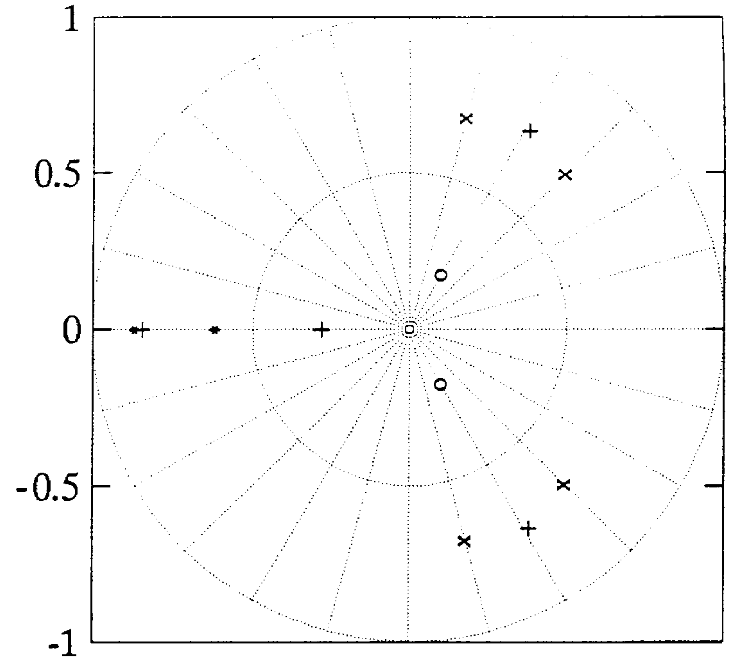
SHORT SEGMENT



ITERATIVE PREFILTERING: ARMA1 DATA

MODEL ORDER $(P, Q) = (4, 2)$



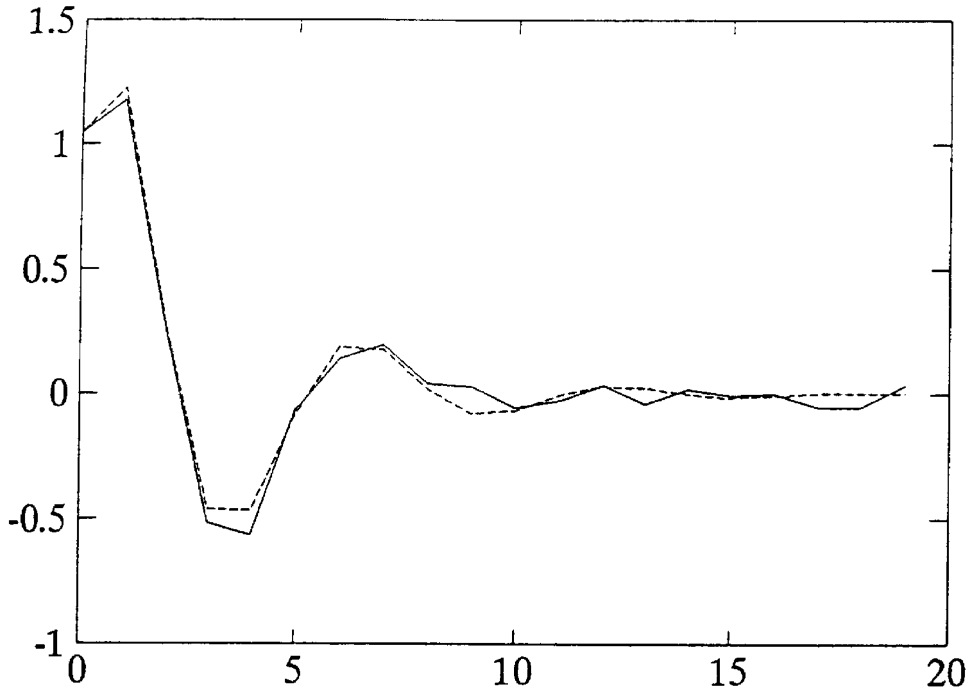


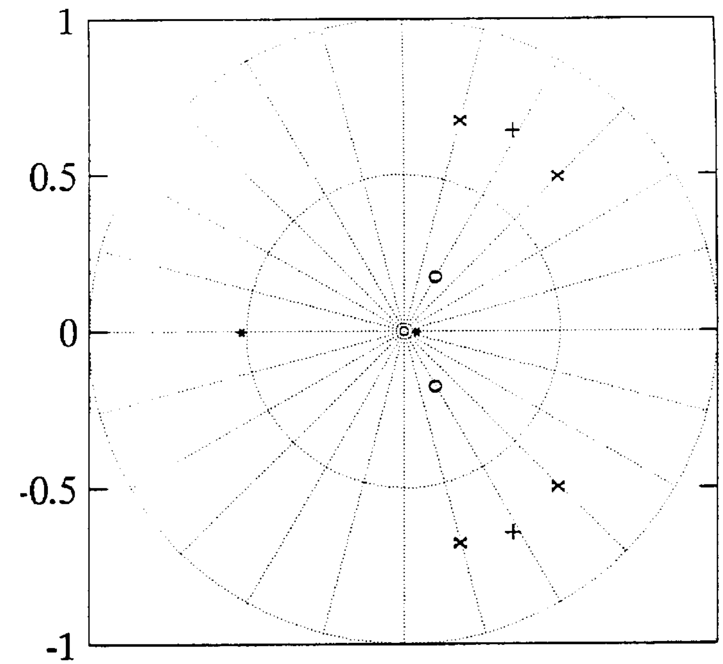
——— Original data
- - - - Model impulse response

$\times$ True poles o True zeros
$+$ Model poles $*$ Model zeros

ITERATIVE PREFILTERING: ARMA1 DATA

REDUCED MODEL ORDER $(P, Q) = (2, 2)$



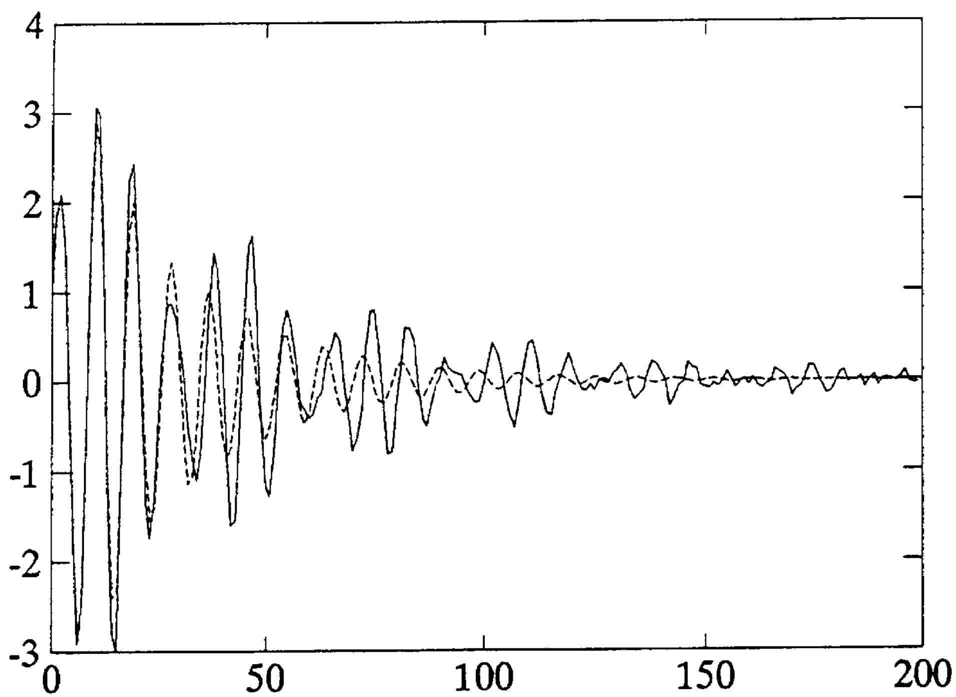


——— Original data
- - - - Model impulse response

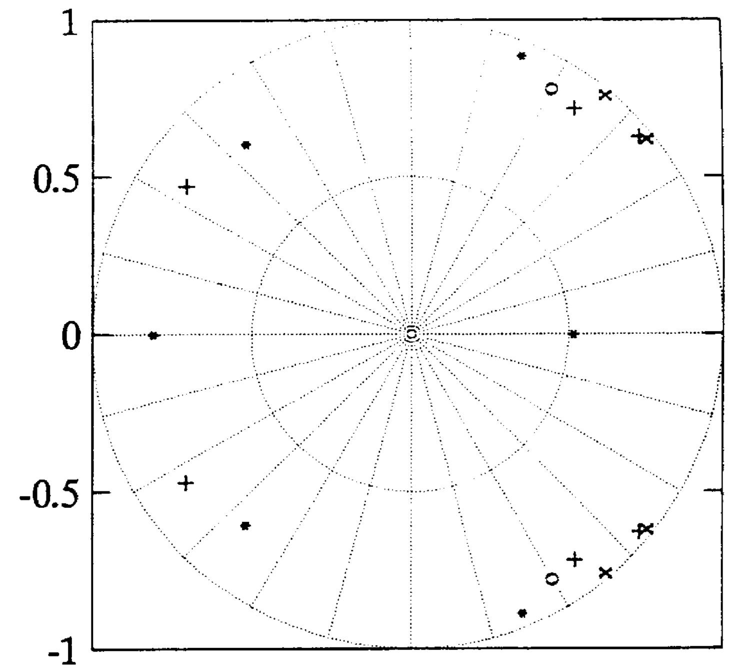
$\times$ True poles
$+$ Model poles
o True zeros
$*$ Model zeros

PRONY'S METHOD: ARMA3 DATA

MODEL ORDER $(P, Q) = (6, 6)$

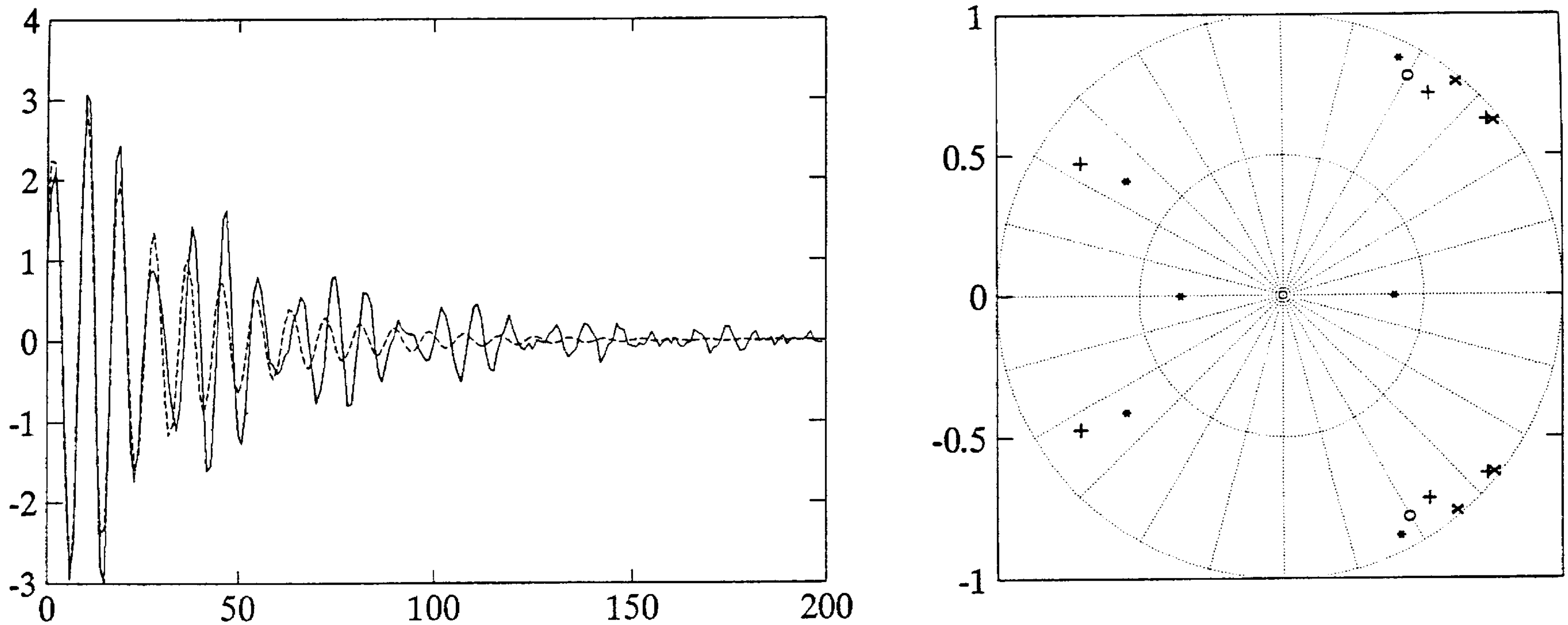


——— Original data
- - - - Model impulse response



$\times$ True poles o True zeros
$+$ Model poles $*$ Model zeros

PRONY'S METHOD: ARMA3 DATA
DURBIN'S METHOD USED TO FIND ZEROS

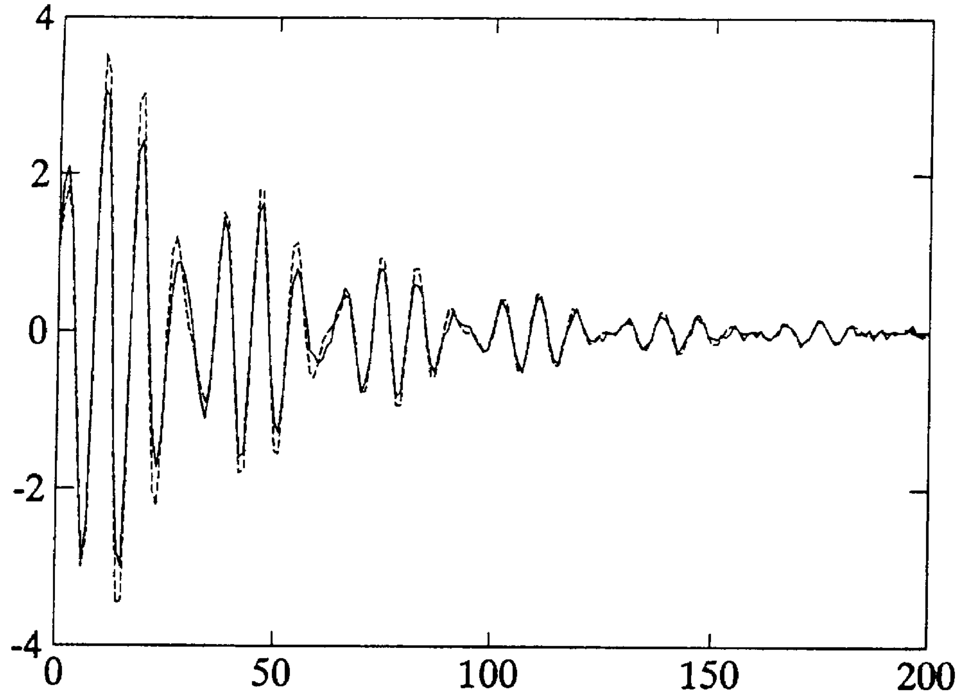


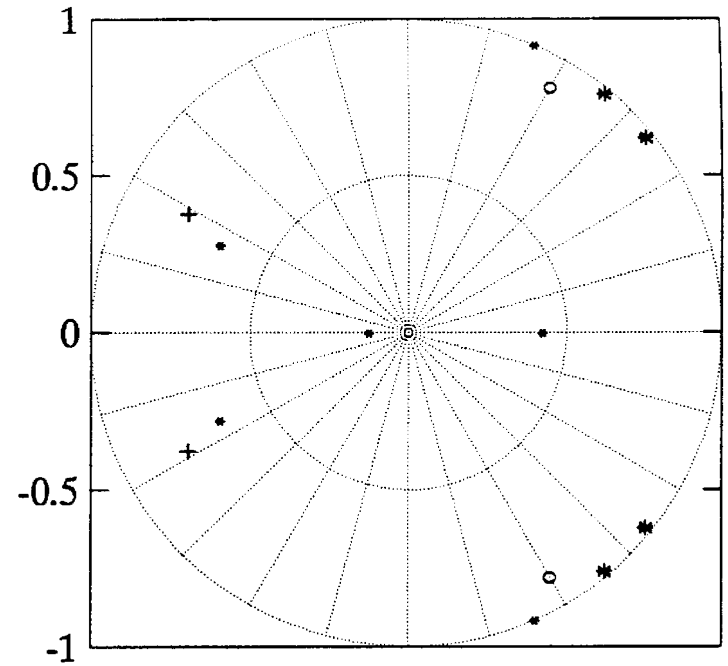
——— Original data
- - - - Model impulse response

× True poles
+ Model poles
o True zeros
∗ Model zeros

LEAST SQUARES YULE-WALKER: ARMA3

MODEL ORDER $(P, Q) = (6, 6)$

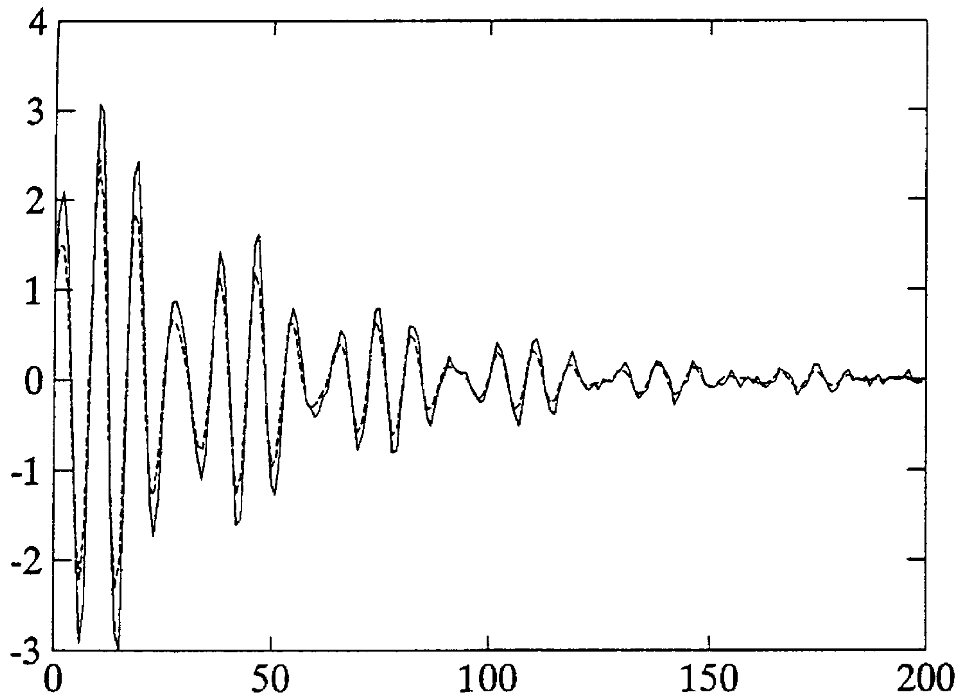




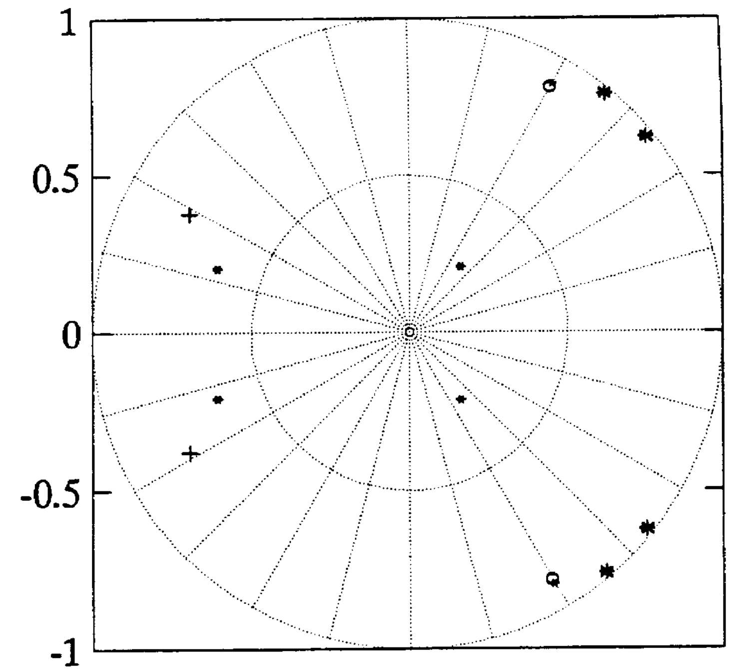
———— Original data
- - - - Model impulse response

$\times$ True poles
$+$ Model poles
o True zeros
$*$ Model zeros

LEAST SQUARES YULE-WALKER: ARMA3
DURBIN'S METHOD USED TO FIND ZEROS



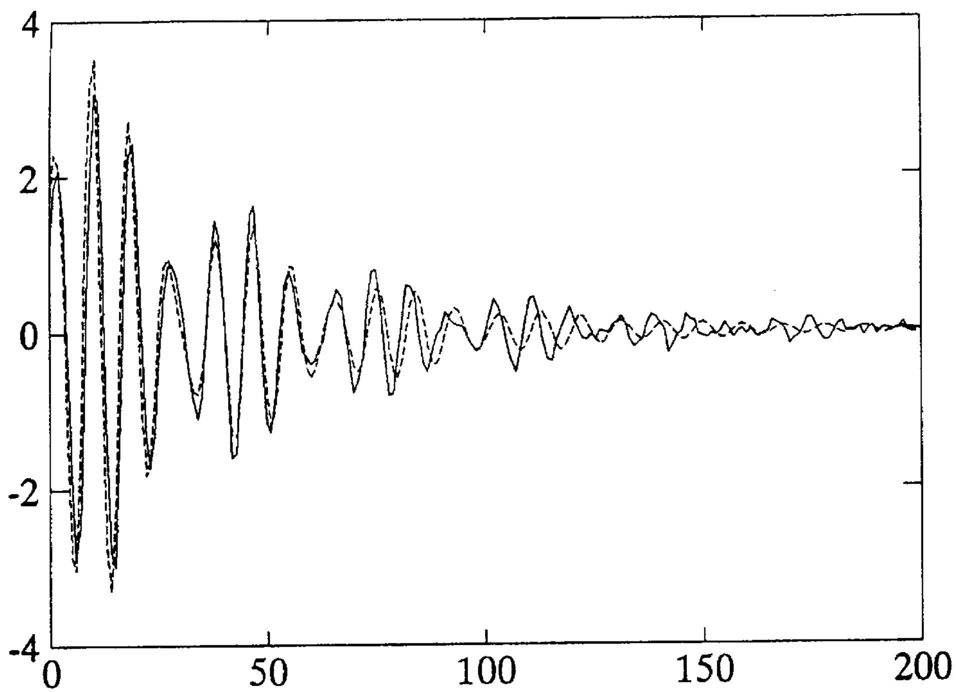
——— Original data
- - - - Model impulse response

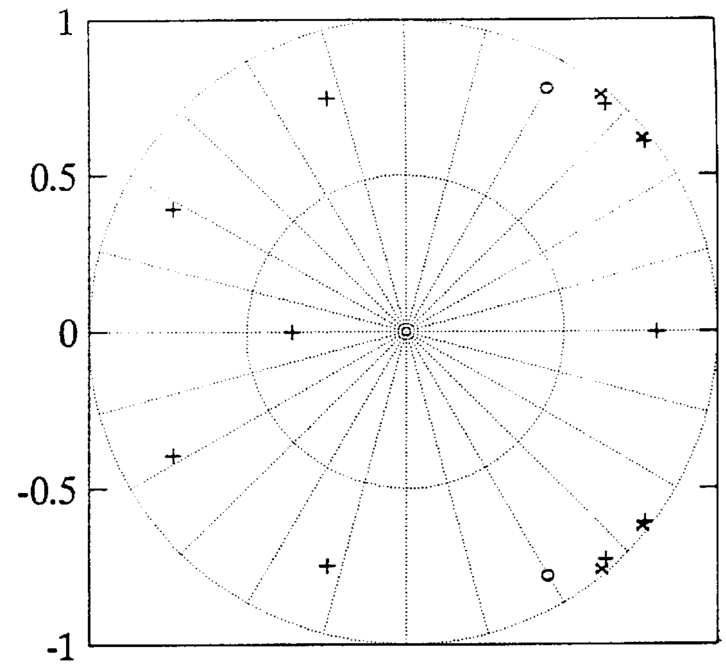


× True poles
o True zeros
+ Model poles
∗ Model zeros

COVARIANCE METHOD: ARMA3 DATA

AR MODEL ORDER $P = 10$





______ Original data
- - - - Model impulse response

$\times$ True poles o True zeros
$+$ Model poles

LEAST SQUARES YULE-WALKER: RULER DATA SHANKS' METHOD USED TO FIND ZEROS

MODEL ORDER $(P, Q) = (6, 16)$